

Integral-free Wigner functions

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Abstract

Wigner phase space quasi-probability distribution function is a Fourier transform related to a given quantum mechanical wave function. It is shown that for the wave functions of type $\psi(q) = e^{-aq^2} \phi(q)$, the Wigner function can be defined in terms of differential operators acting on a given function, independently from the integral formula which appears in the standard definition. Gaussian wave packet, harmonic and singular oscillators are given as the examples.

PACS: 03.65.-w, 03.65.Sq, 05.30.-d

1 Introduction

In a study of quantum corrections to classical statistical mechanics, Wigner constructed the function

$$W(q, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi^*(q - \frac{\hbar}{2}y) \psi(q + \frac{\hbar}{2}y) e^{-iyp} dy. \quad (1)$$

that now called by his name [1]. The goal was to replace the quantum mechanical wave function $\psi(q)$ with a probability distribution in phase space that providing a particlelike description of the underlying wave propagation. Hence the Wigner function (WF) exhibits a representation of quantum states in phase space. Although in quantum mechanics due to the uncertainty principle, a certain point in phase space does not make sense, projection of the WF onto the q and p planes generates the relevant marginal distributions associated with that quantum state, where possible negative values correspond to non-classical situations [2].

WF has been much studied theoretically and experimentally since its introduction, not only in the context of quantum mechanics, but also in various branches of physics and related sciences.

The WF given by (1) can be obtained easily if the integral allow to get an answer analytically. In most cases, despite the existence of wave function $\psi(q)$, the integral becomes quite cumbersome and sometimes it is impossible to handle it.

The aim of the present paper is to present an alternative definition of the WF for the wave functions of a certain type, that is a derivational approach rather than the original integral based one. This new definition gives, at least partly, calculational tricks to overcome the difficulties mentioned above.

The approach considers a class of wave functions given in the form

$$\psi(q) = e^{-aq^2} \phi(q) \quad (a \neq 0), \quad (2)$$

where a is a positive constant, and is based on the series expansion of $\phi(q)$ in term of Hermite polynomials. In order to establish the convergence of the series we need the following theorem [3]: If $\phi(q)$ is a piecewise smooth real function in every finite subinterval $[-b, b]$ and if the integral

$$\int_{-\infty}^{\infty} e^{-q^2} \phi^2(q) dq \quad (3)$$

is finite, then the series

$$\phi(q) = \sum_{n=0}^{\infty} C_n H_n(q), \quad -\infty < q < \infty, \quad (4)$$

where the coefficients C_n can be computed by the orthogonality properties of the Hermite polynomials, converges to $\phi(q)$ at every continuity point of $\phi(q)$. It is also assumed that in the case of complex $\phi(q)$ the consequent of the theorem is still valid *i.e.*, $\phi(q)$ admits a uniformly convergent expansion on the basis of Hermite polynomials.

With these assumptions it is possible to replace the integral by the series and finally to define the WF as the following

$$W(q, p) = \frac{1}{\hbar\sqrt{2\pi a}} e^{-2aq^2} \phi^* \left(q - \frac{i\hbar}{2} \partial_p \right) \phi \left(q + \frac{i\hbar}{2} \partial_p \right) e^{-p^2/2a\hbar^2}, \quad (5)$$

which is the main result of this paper. Note that the assumptions ensure that $\phi(q)$ is smooth enough to be evaluated over the differential operators. It is easy to show that the operators ϕ and ϕ^* in (5) are commutative and their product is real. A useful aspect of this definition is that if $\phi(q)$ contains a finite polynomial alone that may be an orthogonal polynomial such as Hermite or (associated) Laguerre polynomial, then obtaining the WFs for the individual states requires only simple differentiations without needing integral tables or computers.

2 Derivation

With the choice of (2) the WF (1) amounts to

$$W(q, p) = \frac{1}{2\pi} e^{-2aq^2} \int_{-\infty}^{\infty} dy \phi^* \left(q - \frac{\hbar}{2} y \right) \phi \left(q + \frac{\hbar}{2} y \right) e^{-\frac{\hbar^2}{2} ay^2 - iyp} dy. \quad (6)$$

For our purpose, the Hermite polynomials takes the form

$$H_n(q \pm \frac{\hbar}{2} y) = \sum_{r=0}^{[n/2]} \frac{(-1)^r n!}{r! (n-2r)!} 2^{n-2r} (q \pm \frac{\hbar}{2} y)^{n-2r}. \quad (7)$$

By the substitution of (4) and (7) into (6) we get

$$\begin{aligned} W(q, p) &= \frac{1}{2\pi} e^{-2aq^2} \sum_{n,m=0}^{\infty} C_n^* C_m \sum_{r=0}^{[n/2]} \frac{(-1)^r n!}{r! (n-2r)!} 2^{n-2r} \sum_{s=0}^{[m/2]} \frac{(-1)^s m!}{s! (m-2s)!} 2^{m-2s} \\ &\times \int_{-\infty}^{\infty} \left(q - \frac{\hbar}{2} y \right)^{n-2r} \left(q + \frac{\hbar}{2} y \right)^{m-2s} e^{-\frac{\hbar^2}{2} ay^2 - iyp} dy. \end{aligned} \quad (8)$$

If one uses the Binomial expansion for the terms in the integral in (8), the WF takes the form

$$\begin{aligned}
W(q, p) &= \frac{1}{2\pi} e^{-2aq^2} \sum_{n,m=0}^{\infty} C_n^* C_m \sum_{r=0}^{[n/2]} \frac{(-1)^r n!}{r! (n-2r)!} 2^{n-2r} \sum_{s=0}^{[m/2]} \frac{(-1)^s m!}{s! (m-2s)!} 2^{m-2s} \\
&\times \sum_{\alpha=0}^{n-2r} \binom{n-2r}{\alpha} q^{n-2r-\alpha} \left(-\frac{\hbar}{2}\right)^{\alpha} \sum_{\beta=0}^{m-2s} \binom{m-2s}{\beta} q^{m-2s-\beta} \left(\frac{\hbar}{2}\right)^{\beta} \\
&\times \underbrace{\int_{-\infty}^{\infty} y^{\alpha+\beta} e^{-\frac{\hbar^2}{2} ay^2 - iyp} dy}_{\chi(p)},
\end{aligned} \tag{9}$$

where the integral $\chi(p)$ can be adopted to the integral [3]

$$\int_{-\infty}^{\infty} t^n e^{-t^2 + 2itx} dt = \frac{\sqrt{\pi}}{(-i)^n 2^n} e^{-x^2} H_n(x). \tag{10}$$

Thus $\chi(p)$ is straightforward;

$$\chi(p) = (-i)^k \sqrt{\pi} \frac{1}{2^k} \left(\frac{\sqrt{2}}{\hbar\sqrt{a}} \right)^{k+1} e^{-p^2/2a\hbar^2} H_k(p/\hbar\sqrt{2a}), \tag{11}$$

where $k = \alpha + \beta$. If we use the Rodrigues representation of the Hermite polynomials

$$H_k(u) = (-1)^k e^{u^2} \partial_u^k e^{-u^2}, \tag{12}$$

we get $\chi(p)$ as

$$\chi(p) = \frac{\sqrt{2\pi/a}}{\hbar} (i\partial_p)^{\alpha+\beta} e^{-p^2/2a\hbar^2}. \tag{13}$$

Thus the WF yields

$$\begin{aligned}
W(q, p) &= \frac{1}{\hbar\sqrt{2\pi a}} e^{-2aq^2} \sum_{n,m=0}^{\infty} C_n^* C_m \sum_{r=0}^{[n/2]} \frac{(-1)^r n!}{r! (n-2r)!} 2^{n-2r} \sum_{s=0}^{[m/2]} \frac{(-1)^s m!}{s! (m-2s)!} 2^{m-2s} \\
&\times \sum_{\alpha=0}^{n-2r} \binom{n-2r}{\alpha} q^{n-2r-\alpha} \left(-\frac{\hbar}{2}\right)^{\alpha} (i\partial_p)^{\alpha} \\
&\times \sum_{\beta=0}^{m-2s} \binom{m-2s}{\beta} q^{m-2s-\beta} \left(\frac{\hbar}{2}\right)^{\beta} (i\partial_p)^{\beta} e^{-p^2/2a\hbar^2}.
\end{aligned} \tag{14}$$

The last two sums in (14) stand for $(q - \frac{i\hbar}{2}\partial_p)^{n-2r}$ and $(q + \frac{i\hbar}{2}\partial_p)^{m-2s}$ respectively. With this arrangement it is obtained that

$$\begin{aligned}
W(q, p) &= \frac{1}{\hbar\sqrt{2\pi a}} e^{-2aq^2} \sum_{n=0}^{\infty} C_n^* \sum_{r=0}^{[n/2]} \frac{(-1)^r n!}{r! (n-2r)!} [2(q - \frac{i\hbar}{2}\partial_p)]^{n-2r} \\
&\times \sum_{m=0}^{\infty} C_m \sum_{s=0}^{[m/2]} \frac{(-1)^s m!}{s! (m-2s)!} [2(q + \frac{i\hbar}{2}\partial_p)]^{m-2s} e^{-p^2/2a\hbar^2}.
\end{aligned} \tag{15}$$

Thus, with the help of (4) and (7), the Wigner function can finally be compacted as in the equation (5).

3 Applications

3.1 Gaussian wave packet

The wave function for a system represented initially by a Gaussian wave packet is given as

$$\psi(q) = C \exp \left[-\frac{(q - q_0)^2}{4(\Delta q)^2} \right] e^{i p_0 q / \hbar}, \quad (16)$$

where $C = 1/[2\pi(\Delta q)^2]^{1/4}$, Δq is the width of the packet centered at q_0 and p_0 is its average momentum. According to (2)

$$\phi(q) = C \exp \left[-\frac{q_0^2}{4(\Delta q)^2} \right] e^{q_0 q / [2(\Delta q)^2]} e^{i p_0 q / \hbar} \quad (17)$$

and $a = 1/[4(\Delta q)^2]$. Thus

$$\phi^* \left(q - \frac{i\hbar}{2} \partial_p \right) \phi \left(q + \frac{i\hbar}{2} \partial_p \right) = C^2 \exp \left[-\frac{q_0^2}{2(\Delta q)^2} \right] e^{q_0 q / (\Delta q)^2} e^{-p_0 \partial_p}, \quad (18)$$

which is obviously a translation operator which converts p to $p - p_0$. Therefore (5) and the fact that $\Delta q \Delta p = \hbar/2$ yield the WF as

$$W(q, p) = \frac{1}{\pi \hbar} \exp \left[-\frac{(q - q_0)^2}{2(\Delta q)^2} \right] \exp \left[-\frac{(p - p_0)^2}{2(\Delta p)^2} \right] \quad (19)$$

that confirms the correct result.

3.2 Harmonic oscillator

The wave function corresponding to the Hamiltonian $\hat{H} = \hat{p}^2/2 + \hat{q}^2/2$ is given by

$$\psi_n(q) = C_n e^{-q^2/2\hbar} H_n(q/\sqrt{\hbar}), \quad (20)$$

where $C_n = [1/(\pi \hbar)]^{1/4} / \sqrt{2^n n!}$. By virtue of the equation [4]

$$\begin{aligned} & H_n \left[\frac{1}{\sqrt{\hbar}} \left(q - \frac{i\hbar}{2} \partial_p \right) \right] H_n \left[\frac{1}{\sqrt{\hbar}} \left(q + \frac{i\hbar}{2} \partial_p \right) \right] e^{-p^2/\hbar} \\ &= (-1)^n 2^n n! L_n[(2q^2 + 2p^2)/\hbar] e^{-p^2/\hbar} \end{aligned} \quad (21)$$

which is obtained by iteration, (5) gives the well known WF

$$W_n(q, p) = \frac{(-1)^n}{\pi \hbar} e^{-2H/\hbar} L_n(4H/\hbar), \quad (22)$$

where $H = p^2/2 + q^2/2$ and L_n denotes the Laguerre polynomial of order n .

3.3 Singular oscillator

The system is described by the Hamiltonian $\hat{H} = \hat{p}^2/2 + \hat{q}^2/2 + g^2/\hat{q}^2$ with g is a real constant. The normalized eigenfunctions of the Hamiltonian are

$$\psi_n(q) = C_n q^\alpha L_n^{\alpha-1/2}(q^2/\hbar) e^{-q^2/2\hbar}, \quad (23)$$

where $C_n^2 = n!/\Gamma(n+\alpha+1/2) \hbar^{\alpha+1}$, Γ is the Gamma function, $L_n^{\alpha+1/2}$ is the associated Laguerre polynomial and $\alpha = 1/2 + (1/4 + 2g^2)^{1/2}$ [5]. The operator equation version of the WF is straightforward with the help of the argument presented up to now;

$$\begin{aligned} W_n(q, p) &= C_n^2 e^{-q^2/\hbar} \left(q^2 + \frac{\partial_p^2}{4} \right)^\alpha \\ &\times L_n^{\alpha-1/2} \left[\frac{1}{\hbar} \left(q - \frac{i\partial_p}{2} \right)^2 \right] L_n^{\alpha-1/2} \left[\frac{1}{\hbar} \left(q + \frac{i\partial_p}{2} \right)^2 \right] e^{-p^2/\hbar}. \end{aligned} \quad (24)$$

For an arbitrary α and n , integral tables and the approach discussed here fail to get an exact WF of this system. Especially for the fractional values of α , even determining the ground state carries big difficulty since the action of the operator $(q^2 + \partial_p^2/4)^\alpha$ on $\exp(-p^2/\hbar)$ is unknown. (At least, I am not aware of it). But for some special values of α and n , WFs for this system can be obtained explicitly [4].

As a limit of the applicability of the method, consider the one dimensional anyon system [6], where the wave function

$$\psi_n(q) \propto e^{-q^2/2} q^{1/2} H_n(q) \quad (25)$$

corresponds to the Hamiltonian $\hat{H} = \hat{p}^2/2 - 1/\hat{q} - 1/\hat{q}^2$. Obviously the matter arises from the term $q^{1/2}$ since the smoothness condition of $\phi(q)$ is violated.

Acknowledgments

The author wishes to express his appreciations to T. Altanhan and B. S. Kandemir for their helpful discussions. This work was supported in part by the Scientific and Technical Research Council of Turkey (TÜBİTAK).

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